

Accurate Reconstruction of Finite Rate of Innovation Signals on the Sphere

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Abstract—We develop a method for the accurate reconstruction of non-bandlimited finite rate of innovation signals on the sphere. For signals consisting of a finite number of Dirac functions on the sphere, we develop an annihilating filter based method for the accurate recovery of parameters of the Dirac functions using a finite number of observations of the bandlimited signal. In comparison to existing techniques, the proposed method enables more accurate reconstruction primarily due to better conditioning of systems involved in the recovery of parameters. For the recovery of K Diracs on the sphere, the proposed method requires samples of the signal bandlimited in the spherical harmonic (SH) domain at SH degree equal or greater than $K + \sqrt{K + \frac{1}{4}} - \frac{1}{2}$. In comparison to the existing state-of-the art technique, the required bandlimit, and consequently the number of samples, of the proposed method is the same or less. We also conduct numerical experiments to demonstrate that the proposed technique is more accurate than the existing methods by a factor of 10^7 or more for $2 \leq K \leq 20$.

Index Terms—unit sphere, sampling, finite rate of innovation, signal reconstruction, spherical harmonic transform.

EDICS: DSP-FRI, DSP-SPARSE.

I. INTRODUCTION

Development of spherical signal processing techniques finds direct applications in diverse fields of science and engineering where signals are naturally defined on the sphere. These applications include, but not limited to, spherical harmonic lighting in computer graphics [1], signal analysis in diffusion magnetic resonance imaging (dMRI) [2], [3], spectrum estimation in geophysics and cosmology [4], sound analysis and reproduction in acoustics [5], [6] and placement of antennas in wireless communication [7]. To support signal analysis in these applications, accurate reconstruction of signals from a finite number of measurements is inherently required and is therefore of significant importance. In this work, we consider the problem of sampling and accurate reconstruction of non-bandlimited finite rate of innovation (FRI) signal, consisting of finite K number of Dirac delta functions, on the sphere.

Many sampling schemes have been devised in the literature (e.g., see [8] and references therein) for the acquisition of signals bandlimited in the spectral domain, which is enabled by the spherical harmonic (SH) transform – a natural counterpart of the Fourier transform for signals on the

sphere [9]. For the accurate computation of SH transform and accurate reconstruction of a signal bandlimited at SH degree L (formally defined in Section II-A), we require L^2 number of samples [8]. The sampling schemes for taking measurements of bandlimited signals, although permit accurate reconstruction of signal, are not suitable for sampling of non-bandlimited signals such as an ensemble of spikes (Dirac delta functions in the limit) on the sphere which appear in applications in dMRI [3], acoustics and cosmology [6].

Based on the super-resolution theory [10], an algorithm has been developed in [11] for the reconstruction of FRI signals using semi-definite programming, root finding and least-squares. However, the method is iterative in nature and requires Dirac functions to satisfy a minimum separation condition. Recently, following the annihilating filter method devised for signals in one-dimensional Euclidean domain [12] and extended to 2D and higher dimensions [13], signal processing techniques have been proposed [6], [14] for the recovery of parameters of FRI signal on the sphere. The method proposed in [14] requires bandlimiting the FRI signal at $L = 2K$ for the recovery of parameters of Diracs. To reduce the total number of measurements, an alternative reconstruction technique has been developed in [6] which requires the measurements of the FRI signal bandlimited at $L \geq (K + \sqrt{K})$ and therefore reduces the number of samples requirement by a factor of approximately four. For both of these schemes, the error in the reconstruction or recovery of parameters increases with the number of Diracs due to ill-conditioning of the systems required to be inverted during the recovery of parameters.

In this work, we also employ the annihilating filter method in order to develop a method for the recovery of parameters, and consequently accurate reconstruction, of an FRI signal composed of K Diracs on the sphere. Our method requires bandlimiting the signal at $L \geq (K + \sqrt{K + \frac{1}{4}} - \frac{1}{2})$ and therefore takes the same or less¹ number of samples compared to the existing methods. More importantly, in comparison to existing techniques, our method enables more accurate recovery of parameters of FRI signals as we demonstrate through numerical experiments. We organize the rest of the paper as follows. In next section we present the mathematical background and review the existing methods for the problem under consideration. Proposed method is developed in Section III and its analysis is carried out in Section IV. Finally, we make the concluding remarks Section V.

¹Since bandlimit L is required to be an integer, the difference between the bandlimit required by the best of existing algorithms and the proposed method is zero or differs by one.

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II. PRELIMINARIES AND PROBLEM FORMULATION

A. Mathematical Background – Signals on the Sphere

The unit sphere or 2-sphere is defined as $\mathbb{S}^2 = \{\hat{\mathbf{u}} \in \mathbb{R}^3 : |\hat{\mathbf{u}}|_2 = 1\}$, where $|\cdot|_2$ denotes the Euclidean norm and $\hat{\mathbf{u}}$ is the unit vector, parameterized in terms of θ and ϕ as $\hat{\mathbf{u}} \equiv \hat{\mathbf{u}}(\theta, \phi) \triangleq (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)'$. Here $\theta \in [0, \pi]$ is the colatitude angle and $\phi \in [0, 2\pi)$ is the longitude angle. The inner product between two functions f and g on the sphere is defined as

$$\langle f, g \rangle \triangleq \int_{\mathbb{S}^2} f(\hat{\mathbf{u}}) \overline{g(\hat{\mathbf{u}})} ds(\hat{\mathbf{u}}), \quad (1)$$

where $ds(\hat{\mathbf{u}}) = \sin \theta d\theta d\phi$ is the differential area element on \mathbb{S}^2 , $\overline{(\cdot)}$ denotes the complex conjugate and the integration is carried out over the entire sphere. The complex-valued functions on the 2-sphere form a Hilbert space $L^2(\mathbb{S}^2)$ equipped with the inner product defined in (1). The inner product defined in (1) induces a norm $\|f\| \triangleq \langle f, f \rangle^{1/2}$. We refer to the functions with finite induced norm as signals on the sphere.

For the space $L^2(\mathbb{S}^2)$, spherical harmonic functions (or spherical harmonics for short) serve as complete orthonormal basis and are defined as [9]

$$Y_\ell^m(\hat{\mathbf{u}}) \equiv Y_\ell^m(\theta, \phi) \triangleq \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}, \quad (2)$$

for integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$. Here, P_ℓ^m is the associated Legendre function of degree ℓ and order m and is given by [9]

$$P_\ell^m(\nu) = \frac{(-1)^m}{2^\ell \ell!} (1-\nu^2)^{m/2} \frac{d^{\ell+m}}{d\nu^{\ell+m}} (\nu^2-1)^\ell \quad (3)$$

$$P_\ell^{-m}(\nu) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\nu), \quad (4)$$

for $0 \leq m \leq \ell$ and $|\nu| \leq 1$. Due to completeness of spherical harmonics, we can represent any signal $f \in L^2(\mathbb{S}^2)$ as

$$f(\hat{\mathbf{u}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_\ell^m Y_\ell^m(\hat{\mathbf{u}}), \quad (5)$$

where $(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle$ [9] denotes the SH coefficient of integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$. The spherical harmonic coefficients form the representation of a signal in spectral (Fourier) domain. We define the function f to be bandlimited in spectral domain at degree L if $(f)_\ell^m = 0, \forall \ell \geq L, -\ell \leq m \leq \ell$.

B. Problem under Consideration

We consider a signal consisting of K Diracs on the sphere given by

$$f(\hat{\mathbf{u}}) = \sum_{k=1}^K \alpha_k \delta(\hat{\mathbf{u}}, \hat{\mathbf{u}}_k), \quad (6)$$

where α_k is the complex amplitude and $\hat{\mathbf{u}}_k \equiv \hat{\mathbf{u}}_k(\theta_k, \phi_k)$ represents the location of k -th Dirac on the sphere. Here $\delta(\hat{\mathbf{u}}, \hat{\mathbf{u}}_k)$ is the Dirac delta function defined on the sphere which, similar to its linear counterpart, is identified by its

sifting property $\langle f, \delta(\cdot, \hat{\mathbf{u}}_k) \rangle = f(\hat{\mathbf{u}}_k)$. The problem under consideration is to accurately recover the amplitudes α_k and locations $\hat{\mathbf{u}}_k$ of K Diracs of the signal f , given the samples of f bandlimited in the spectral domain.

C. Review of Existing Methods

We here review the existing methods presented in literature [6], [14] for the recovery of parameters of signal of the form given in (6). Recently, an algorithm has been presented in [14], based on the annihilation filter method [12], for the recovery of the parameters f which requires the computation of spherical harmonic coefficients $(f)_\ell^m$ of f for degrees $\ell < 2K$ and orders $|m| \leq \ell$, which are computed by first convolving the signal f with a sampling kernel which bandlimits the signal at degree $L = 2K$. If the recently proposed optimal-dimensionality sampling [8] is employed for the computation of SH coefficients, the method requires L^2 samples of the signal f bandlimited at $L = 2K$. Employing the SH coefficients, the method then forms a Toeplitz system which enables the computation of ϕ_k using which α_k and θ_k are recovered. The method assumes that $\theta_k \notin \{0, \pi\}$ and $\theta_j \neq \pi - \theta_k$ when $\phi_j = \phi_k$ for $j, k = 1, 2, \dots, K$ and $j \neq k$.

To reduce the number of samples required for the recovery of parameters, an algorithm has been presented more recently in [6], which requires SH coefficients $(f)_\ell^m$ for $\ell \leq L$ and orders $|m| \leq \ell$ of the signal f bandlimited at SH degree² $L = \lceil K + \sqrt{K} \rceil$. Consequently, when compared to the method in [14], this method requires (approximately) *four* times less number of samples. The SH coefficients are then used to form an annihilating matrix [12] which enables the computation of θ_k , which are then used to recover the parameters α_k and ϕ_k . The algorithm works only when $\theta_k \notin \{0, \pi\}$ and $\theta_j \neq \theta_k$ for $j \neq k$ and $j, k = 1, 2, \dots, K$. Although both of these methods allow recovery of parameters, the reconstruction error increases with the increase in number of Diracs on the sphere as we illustrate later in the paper.

III. ACCURATE RECONSTRUCTION OF SIGNALS WITH FRI

Here, we propose a method for the recovery of the parameters of f given in (6). In comparison to the existing methods [6], [14], the proposed algorithm has significantly smaller reconstruction error. Furthermore, our method requires (approximately) four times less and the same (or less) number of samples than those required by the methods presented in [14] and [6], respectively.

A. Formulation

By employing the sifting property of Dirac delta function, we can express the SH coefficient $(f)_\ell^m = \langle f, Y_\ell^m \rangle$ of f given in (6) as

$$(f)_\ell^m = \sum_{k=1}^K \alpha_k \overline{Y_\ell^m(\theta_k, \phi_k)}. \quad (7)$$

²Here $\lceil \cdot \rceil$ denotes the integer ceiling function.

which, noting that $Y_\ell^m(\theta, \phi) = Y_\ell^m(\theta, 0) e^{im\phi}$ (using (2)) and $Y_\ell^m(\theta, 0)$ is a product of $(\sin \theta)^{|m|}$ and a polynomial in $\cos \theta$ of degree $(\ell - |m|)$ (using (3)), can be expressed as

$$(f)_\ell^m = \sum_{k=1}^K \alpha_k \sum_{p=0}^{\ell-|m|} c_{\ell m}^p (\cos \theta_k)^p (\sin \theta_k)^{|m|} e^{-im\phi_k}, \quad (8)$$

where $c_{\ell m}^p$ denotes the coefficient associated with $(\cos \theta)^p$ of the polynomial defining $Y_\ell^m(\theta, 0)$. For order m , we rearrange (8) to get,

$$(f)_\ell^m = \sum_{p=0}^{\ell-|m|} c_{\ell m}^p d_{pm}, \quad (9)$$

where

$$d_{pm} = \begin{cases} \sum_{k=1}^K (\alpha_k y_{kp}) x_k^m & 0 \leq m < L, \\ \sum_{k=1}^K (\alpha_k y_{kp}) \overline{x_k^m} & -L < m < 0, \end{cases} \quad (10)$$

with $x_k = \sin \theta_k e^{-i\phi_k}$ and $y_{kp} = (\cos \theta_k)^p$. Clearly, both d_{pm} for $0 \leq m < L$ and $\overline{d_{pm}}$ for $-L < m < 0$ are linear combination of exponentials x_k^m and therefore are of special interest as the annihilating filter technique [12] can be used to recover x_k .

B. Recovery of Longitudes of Diracs

We consider that the measurements of the signal f bandlimited at degree L are available such that the spherical harmonic coefficients $(f)_\ell^m$ can be accurately computed for all degrees $\ell < L$ and all orders $|m| \leq \ell$. We shortly present the bandlimit L required for the accurate recovery of parameters.

In (9), $(f)_\ell^m$ for $|m| \leq \ell < L$ and d_{pm} for $0 \leq p < L - |m|$ form a linear system of equations for each $|m| < L$ with triangular coefficient matrix of size $(L - |m|) \times (L - |m|)$. Consequently, d_{pm} for $0 \leq p < L - |m|$ can be recovered exactly for each $|m| < L$ using (9).

Once d_{pm} is computed, we employ the annihilating filter technique [12] to estimate x_k as d_{pm} is a linear combination of K powers of x_k . This estimation involves the construction of annihilating matrix \mathbf{Z} given by

$$\mathbf{Z} = \begin{bmatrix} d_{0,L-1} & d_{0,L-2} & \cdots & d_{0,L-K-1} \\ d_{0,L-2} & d_{0,L-3} & \cdots & d_{0,L-K-2} \\ \vdots & \vdots & \ddots & \vdots \\ d_{0,K} & d_{0,K-1} & \cdots & d_{0,0} \\ \hline \overline{d_{0,-(L-1)}} & \overline{d_{0,-(L-2)}} & \cdots & \overline{d_{0,-(L-K-1)}} \\ \overline{d_{0,-(L-2)}} & \overline{d_{0,-(L-3)}} & \cdots & \overline{d_{0,-(L-K-2)}} \\ \vdots & \vdots & \ddots & \vdots \\ \hline \overline{d_{0,-(K)}} & \overline{d_{0,-(K-1)}} & \cdots & \overline{d_{0,0}} \\ d_{1,L-2} & d_{1,L-3} & \cdots & d_{1,L-K-2} \\ d_{1,L-3} & d_{1,L-4} & \cdots & d_{1,L-K-3} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,K} & d_{1,K-1} & \cdots & d_{1,0} \\ \hline \overline{d_{1,-(L-2)}} & \overline{d_{1,-(L-3)}} & \cdots & \overline{d_{1,-(L-K-2)}} \\ \overline{d_{1,-(L-3)}} & \overline{d_{1,-(L-4)}} & \cdots & \overline{d_{1,-(L-K-3)}} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}, \quad (11)$$

followed by the computation of its right singular vector \mathbf{v} .

Lemma 1. *If the longitudes and colatitudes of K Diracs of the signal f are such that $\theta_j \neq \pi - \theta_k$ when $\phi_j = \phi_k$ for $j, k = 1, 2, \dots, K$, $j \neq k$, the null-space, denoted by $\mathcal{N}(\mathbf{Z})$, of the annihilating matrix \mathbf{Z} with at least K rows is 1-dimensional.*

Proof: If $\theta_j \neq \pi - \theta_k$ when $\phi_j = \phi_k$ for $j, k = 1, 2, \dots, K$, $j \neq k$, each $x_k = \sin \theta_k e^{-i\phi_k}$ corresponding to the k -th Dirac is unique. From (10), we note that the rank of \mathbf{Z} with at least K rows is equal to K when all x_k are unique. Consequently, $\mathcal{N}(\mathbf{Z})$ is 1-dimensional. ■

Remark 1 (On the Recovery of x_k). *Now we employ the annihilating filter property to estimate x_k using $\mathbf{v} \in \mathcal{N}(\mathbf{Z})$. A finite impulse response (FIR) filter is known as annihilating filter if zeros of the filter are placed such that the filter annihilates the signal. Since $\mathbf{v} \in \mathcal{N}(\mathbf{Z})$, we have*

$$\mathbf{d}_q^T \mathbf{v} = 0, \quad (12)$$

for any $\mathbf{d}_q \triangleq \mathbf{Z}\{q, : \}$, that is, the q -th row of the annihilating matrix \mathbf{Z} . Consequently, \mathbf{v} is a vector of coefficients of the FIR filter which annihilates the signal of the form (10). The transfer function of such annihilating filter is given by

$$V(z) \triangleq \prod_{k=1}^K (1 - x_k z^{-1}) \triangleq \sum_{n=0}^K v_n z^{-n}. \quad (13)$$

Since we have determined \mathbf{v} , we obtain the estimate of x_k , $k = 1, 2, \dots, K$ by finding the roots, denoted by \tilde{x}_k , $k = 1, 2, \dots, K$, of the annihilating filter.

Using \tilde{x}_k and noting that $x_k = \sin \theta_k e^{-i\phi_k}$, we recover the longitudes ϕ_k from \tilde{x}_k as

$$\tilde{\phi}_k = -\text{Phase}\{\tilde{x}_k\}, \quad (14)$$

where $\text{Phase}\{\cdot\}$ returns the phase of the complex argument.

Remark 2 (On the Bandlimit Requirement). *For a signal f bandlimited at L , the maximum number of rows of \mathbf{Z} which can be constructed is $2 \times (L - K) + 2 \times (L - K - 1) + \cdots + 2 \times 2 + 2 \times 1$. Following Lemma 1, we require matrix \mathbf{Z} to have at least K rows to ensure a unique vector $\mathbf{v} \in \mathcal{N}(\mathbf{Z})$. Consequently, we require*

$$L \geq K + \sqrt{K + \frac{1}{4}} - \frac{1}{2} \quad (15)$$

C. Recovery of Colatitudes and Amplitudes of Diracs

Here we use the estimated \tilde{x}_k to recover colatitude θ_k and amplitude α_k for $k = 1, \dots, K$. For $p = 0$, we have d_{pm} for $m = 0, 1, \dots, L - 1$, which we explicitly rewrite, using (10), as

$$d_{0m} = \sum_{k=1}^K \alpha_k x_k^m. \quad (16)$$

Since we have chosen $L > K$ (Remark 2), we can form the following Vandermonde system using (10):

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_K \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{K-1} & x_2^{K-1} & \cdots & x_K^{K-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_K \end{bmatrix} = \begin{bmatrix} d_{00} \\ d_{01} \\ \vdots \\ d_{0K-1} \end{bmatrix}. \quad (17)$$

Provided x_k , $k = 1, 2, \dots, K$, are distinct as ensured by Lemma 1, the Vandermonde system above enables recovery of amplitudes $\tilde{\alpha}_k$, $k = 1, 2, \dots, K$.

To recover colatitude parameter, we use d_{pm} for $p = 1$ and $m = 0, 1, \dots, L - 2$. Using d_{1m} , given in (10), and noting that $y_{k1} = \cos \theta_k$, we formulate another Vandermonde system given by

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_K \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{K-1} & x_2^{K-1} & \cdots & x_K^{K-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \cos \theta_1 \\ \alpha_2 \cos \theta_2 \\ \vdots \\ \alpha_K \cos \theta_K \end{bmatrix} = \begin{bmatrix} d_{10} \\ d_{11} \\ \vdots \\ d_{1K-1} \end{bmatrix}. \quad (18)$$

The solution of above system yields an estimate of $\alpha_k \cos \theta_k$, denoted by $\mathbb{E}\{\alpha_k \cos \theta_k\}$, for $k = 1, 2, \dots, K$. Since we have already recovered amplitude as $\tilde{\alpha}_k$, we recover the colatitude as

$$\tilde{\theta}_k = \arccos \left[\frac{\mathbb{E}\{\alpha_k \cos \theta_k\}}{\tilde{\alpha}_k} \right], \quad k = 1, 2, \dots, K. \quad (19)$$

IV. ANALYSIS

Here we compare the bandlimit requirement and reconstruction accuracy of the proposed method with the existing methods [6], [14]. We corroborate, through numerical experiments, the claim that the proposed method is superior in terms of accuracy of recovered parameters in comparison with the existing methods.

A. Bandlimit Requirement

As mentioned earlier, we require L^2 number of measurements of the signal bandlimited at L to compute its spherical harmonic coefficients. Consequently, it is desirable for a method to have smaller bandlimit requirements to reduce the total number of measurements. For the recovery of parameters of the signal consisting of K Diracs on the sphere, the bandlimit required by the proposed algorithm is $L = \lceil K + \sqrt{K + \frac{1}{4} - \frac{1}{2}} \rceil$, which is much smaller as compared to $L = 2K$ required by the method in [14] and the same (or less) than $L = \lceil K + \sqrt{K} \rceil$ required for the algorithm presented in [6].

B. Accuracy Analysis

In order to compare the recovery/reconstruction error of the proposed method with the algorithms presented in [14] and [6], we implement each method in MATLAB³ and recover the parameters of the signal f , of the form given in (6), by

³For our method, we make the code publicly available at <http://zubairkhalid.org/fri> to facilitate the reproduction of research results.

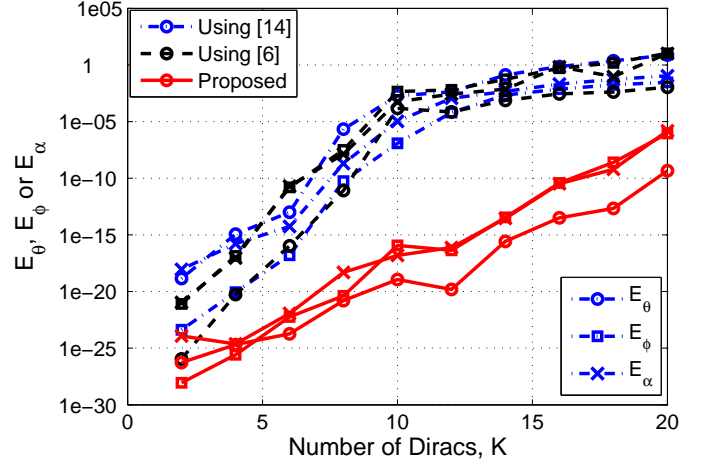


Fig. 1: Mean Error E_θ , E_ϕ and E_α between recovered and original colatitudes, longitudes and amplitudes respectively for different values of $2 \leq K \leq 20$ (number of Diracs) using the existing techniques [6], [14] and the proposed algorithm.

conducting following experiment. For each $K = 2, 4, \dots, 20$, we randomly choose the parameters⁴ $\theta_k \in (0, \pi)$, $\phi_k \in [0, 2\pi)$ and α_k with real and imaginary parts uniformly distributed in $[-1, 1]$ for $k = 1, 2, \dots, K$. For each method and each K , we recover the parameters $\tilde{\theta}_k$, $\tilde{\phi}_k$ and $\tilde{\alpha}_k$ of the signal and compute the mean-squared errors given by $E_\theta = \frac{1}{K} \sum_{k=1}^K |\tilde{\theta}_k - \theta_k|^2$,

$E_\phi = \frac{1}{K} \sum_{k=1}^K |\tilde{\phi}_k - \phi_k|^2$, $E_\alpha = \frac{1}{K} \sum_{k=1}^K |\tilde{\alpha}_k - \alpha_k|^2$. We plot these errors, averaged over 1000 trials of the experiment, in Fig. 1, where it is evident that the proposed method enables more accurate recovery of parameters when compared to other methods in literature. On average, the proposed algorithm outperforms the other methods in terms of smaller recovery error by a factor up to 10^7 .

V. CONCLUSIONS

In this work, we have proposed a method for accurate reconstruction of an FRI signal consisting of K Dirac functions on the sphere. The proposed method takes samples of the signal bandlimited in the SH domain at the SH degree $L = \lceil K + \sqrt{K + \frac{1}{4} - \frac{1}{2}} \rceil$ for the computation of SH transform of the signal. Following the computation of SH coefficients, we recover the parameters of the Diracs using the annihilating filter method, root finding and solving a series of linear systems. The proposed method requires the same or less number of samples compared to the best of existing methods. More importantly, the error in the recovery of parameters is significantly smaller.

⁴The parameters are randomly generated such that we have K distinct θ_k and $x_k = \sin \theta_k e^{-i\phi_k}$ as the method in [6] requires θ_k , $k = 1, 2, \dots, K$ to be unique, whereas the method in [14] and our proposed method require x_k , $k = 1, 2, \dots, K$ to be unique (Lemma 1). This is avoided by imposing the condition that the K Diracs have at least $\pi/3K$ distance among them.

REFERENCES

- [1] L. Zhang and D. Samaras, "Face recognition from a single training image under arbitrary unknown lighting using spherical harmonics," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 28, no. 3, pp. 351–363, Mar. 2006.
- [2] A. P. Bates, Z. Khalid, and R. A. Kennedy, "An optimal dimensionality sampling scheme on the sphere with accurate and efficient spherical harmonic transform for diffusion MRI," *IEEE Signal Process. Lett.*, vol. 23, pp. 15–19, January 2016.
- [3] S. Deslauriers-Gauthier and P. Marziliano, "Spherical finite rate of innovation theory for the recovery of fiber orientations," in *Proc. IEEE Eng. Med. Biol. Soc.*, Aug. 2012, pp. 2294–2297.
- [4] F. A. Dahlen and F. J. Simons, "Spectral estimation on a sphere in geophysics and cosmology," *Geophys. J. Int.*, vol. 174, pp. 774–807, 2008.
- [5] A. P. Bates, Z. Khalid, and R. A. Kennedy, "Novel sampling scheme on the sphere for head-related transfer function measurements," *IEEE/ACM Trans. Audio Speech Language Process.*, vol. 23, pp. 1068–1081, June 2015.
- [6] I. Dokmani and Y. M. Lu, "Sampling sparse signals on the sphere: Algorithms and applications," *IEEE Trans. Signal Process.*, vol. 64, no. 1, pp. 189–202, Jan 2016.
- [7] Y. F. Alem, Z. Khalid, and R. A. Kennedy, "3D spatial fading correlation for uniform angle of arrival distribution," *IEEE Commun. Lett.*, vol. 19, pp. 1073–1076, June 2015.
- [8] Z. Khalid, R. A. Kennedy, and J. D. McEwen, "An optimal-dimensionality sampling scheme on the sphere with fast spherical harmonic transforms," *IEEE Trans. Signal Process.*, vol. 62, no. 17, pp. 4597–4610, Sept 2014.
- [9] R. A. Kennedy and P. Sadeghi, *Hilbert Space Methods in Signal Processing*. Cambridge, UK: Cambridge University Press, Mar. 2013.
- [10] E. J. Candès and C. Fernandez-Granda, "Towards a mathematical theory of super-resolution," *Comm. Pure Appl. Math.*, vol. 67, no. 6, pp. 906–956, Jun. 2014.
- [11] T. Bendory, S. Dekel, and A. Feuer, "Super-resolution on the sphere using convex optimization," *IEEE Trans. Signal Process.*, vol. 63, no. 9, pp. 2253–2262, May 2015.
- [12] M. Vetterli, P. Marziliano, and T. Blu, "Sampling signals with finite rate of innovation," *IEEE Trans. Signal Process.*, vol. 50, no. 6, pp. 1417–1428, Jun 2002.
- [13] P. Shukla and P. L. Dragotti, "Sampling schemes for multidimensional signals with finite rate of innovation," *IEEE Trans. Signal Process.*, vol. 55, no. 7, pp. 3670–3686, Jul. 2007.
- [14] S. Deslauriers-Gauthier and P. Marziliano, "Sampling signals with a finite rate of innovation on the sphere," *IEEE Trans. Signal Process.*, vol. 61, no. 18, pp. 4552–4561, Sept 2013.